

Equilibrium model of rational and noise traders: bifurcations to endogenous bubbles

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Abstract: We introduce a model of financial bubbles with two assets (risky and risk-less), in which rational investors and noise traders co-exist. Rational investors form continuously evolving expectations on the return and risk of a risky asset and maximize their expected utility with respect to their allocation on the risky asset versus the risk-free asset. Noise traders are subjected to social imitation and follow momentum trading. We find the existence of a set of bifurcations controlled by the relative influence of noise traders with respect to rational investors that separate a normal regime of the price dynamics to a phase punctuated by recurrent exponentially explosive bubbles. The transition to a bubble regime is favored by noise traders who are more social, and who use more momentum trading with shorter time horizons. The model accounts well for the behavior of traders and for the price dynamics that developed during the dotcom bubble in 1995-2000. Momentum strategies are shown to be transiently profitable, supporting these strategies as enhancing herding behavior.

keywords: noise traders, financial bubbles, bifurcation, momentum trading, dotcom bubble, arbitrage

JEL:

C73 - Stochastic and Dynamic Games; Evolutionary Games; Repeated Games

G01 - Financial crises

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1 Introduction

The very existence of financial bubbles has been a controversial and elusive subject. Some have argued that financial bubbles play a huge role in the global economy, affecting hundreds of millions of people (Kindleberger, 1978; Shiller, 2000; Sornette, 2003). Others have basically ignored or refuted their possibility (Fama, 1998). Moreover, until recently, the existence of such bubbles, much less their effects, have been ignored at the policy level. Finally, only after the most recent historical global financial crisis, officials at the highest level of government and academic finance have acknowledged the existence and importance of identifying and understanding bubbles. The President of the Federal Reserve Bank of New York, William C. Dudley, stated in April 2010 “what I am proposing is that we try—try to identify bubbles in real time, try to develop tools to address those bubbles, try to use those tools when appropriate to limit the size of those bubbles and, therefore, try to limit the damage when those bubbles burst.” Such a statement from the New York Fed representing, essentially, the monetary policy of the United States governmental banking system would have been, and, in some circles, still is, unheard of. This, in short, is a bombshell and a wake-up call to academics and practitioners. Dudley exhorts to try to develop tools to address bubbles.

But before acting against bubble, before even making progress in ex-ante diagnosing bubbles before, one needs to define what is a bubble. The problem is that the “econometric detection of asset price bubbles cannot be achieved with a satisfactory degree of certainty. For each paper that finds evidence of bubbles, there is another one that fits the data equally well without allowing for a bubble. We are still unable to distinguish bubbles from time-varying or regime-switching fundamentals, while many small sample econometrics problems of bubble tests remain unresolved.” summarizes Gurkaynak (2008) in his review paper.

Let us start with the rather generally accepted stylized fact that, in a period where a bubble is present, the stock return exhibits transient excess return above the long-term historical average, giving rise to what could be termed a “bubble risk premium puzzle”. For instance, as we report in the empirical section, the valuation of the Internet stock index went from a reference value 100 in January 1998 to a peak of 1400.06 in March 9, 2000, corresponding to an annualized return of more than 350% ! A year and a half later, the Internet stock valuation was back at its pre-1998 level. Another stylized fact well represented during the dotcom bubble is the highly intermittent or punctuated growth of the stock prices, with super-exponential accelerations followed by transient corrections, themselves followed by further vigorous re-

bounds (Johansen and Sornette, 2010; Sornette and Woodard, 2010). Bubbles are usually followed by crashes, in an often tautological logic resulting from the fact that the existence of a crash is usually taken as the ex-post signature of the bubble, as summarized by A. Greenspan (2002): “We, at the Federal Reserve... recognized that, despite our suspicions, it was very difficult to definitively identify a bubble until after the fact, that is, when its bursting confirmed its existence...” More optimistically but still controversial, recent systematic econometric studies have shown that it is possible to relate objectively an anomalous transient excess return and the subsequent crash (Sornette, 2003; Johansen and Sornette, 2010; Sornette et al., 2011).

The recent finance literature has evolved to increasingly recognize the evidence of bubbles which is defined as deviations from fundamental value. One important class of theories shows that there can be large movements in asset prices due to the combined effects of heterogeneous beliefs and short-sales constraints. The basic idea finds its root back to the Lintner (1969)’s CAPM model of asset prices with investors having heterogeneous beliefs. In his model, asset prices are a weighted average of beliefs about asset payoffs. Lintner (1969) demonstrates that widely inflated prices can occur in his model. Many asset pricing models in the spirit of Lintner (1969) have been proposed (See, for example, Miller (1977), Jarrow (1980), Harrison and Kreps (1978), Chen, Hong and Stein (2000), Scheinkman and Xiong (2003) and Duffie, Garleanu and Pedersen (2002)). In these models which assume heterogeneous beliefs and short sales restrictions, the asset prices are determined at equilibrium to the extent that they reflect the heterogeneous beliefs about payoffs, but short sales restrictions force the pessimistic investors out of the market, leaving only optimistic investors and thus inflated asset price levels. However, when short sales restrictions no longer bind investors, then prices fall back down. This provides a possible account of the bursting of the Internet bubble that developed in 1998-2000. As documented in Ofek and Richardson (2003) and Cochrane (2003), approximately 80 percent of Internet-related shares were locked up, so that the float of the Internet sector dramatically increased as the lockups of many of these stocks expired in Spring 2000. The reduction of short sale constraints, by the dual process of expiration of lockups and expansion in the number of Internet companies with publicly tradable shares, constitutes an important component in the collapse of Internet stock prices. Among many, Hong, Scheinkman and Xiong (2006) model explicitly the relationship between the number of publicly tradable shares of an asset and the propensity for speculative bubbles to form.

Another extensive body of literature is devoted to *noise-trader* (also referred to as positive-feedback investors). The term “noise traders” was in-

troduced first by Kyle (1985) and Black (1986) to describe irrational investors. Thereafter, many scholars exploited this concept to extend the standard models by introducing the simplest possible heterogeneity in terms of two interacting populations of rational and irrational agents. One can say that the one-representative-agent theory is being progressively replaced by a two-representative-agents theory, analogously to the progress from the one-body to the two-body problems in Physics. It has been often explained that markets bubble and crash in the absence of significant shifts in economic fundamentals is often explained to occur when herders such as noise-traders deliberately act against their private information and follow the crowd.

De Long, Shleifer, Summers and Waldmann (1990a, 1990b) proposed the first model of market bubbles and crashes which exploits this idea of the possible role of noise traders following positive feedback strategies or momentum investment strategies in the development of bubbles. They showed a possible mechanism for why asset prices may deviate from the fundamentals over long time periods. The key point is that trading between rational arbitrageurs and noise-traders gives rise to bubble-like price patterns. In their model, rational speculators destabilizes prices because their trading triggers positive feedback trading by noise-traders. Positive feedback trading by noise-traders leads to a positive auto-correlation of returns at short horizons. Eventually, arbitrage by rational speculators will pull the prices back to fundamentals. Their arbitrage trading leads to a negative autocorrelation of returns at longer horizons.

Their work was followed by a number of empirical studies on positive feedback tradings. The influential empirical evidence on positive feedback trading came from the works of De Bondt and Thaler (1985), and Jegadeesh and Titman (1993, 2001), which established that stock returns exhibit momentum behavior at intermediate horizons, and reversals at long horizons. That is, strategies which buy stocks that have performed well in the past and sell stocks that have performed poorly in the past generate significant positive returns over 3- to 12- month holding periods. However, stocks that perform poorly in the past perform better over the next 3 to 5 years than stocks that perform well in the past. Behavioral models that explain the coexistence of intermediate horizon momentum and long horizon reversals in stock returns are proposed by Barberis, Shleifer, and Vishny (1998), Daniel, Hirshleifer, and Subrahmanyam (1998), and Hong and Stein (1999).

Abreu and Brunnermeier (2003) propose a different mechanism justifying why rational traders ride rather than arbitrage bubbles under the condition in which the stock price is kept above its fundamental value by irrationally exuberant behavioral traders such as noise-traders. They consider a market where arbitrageurs face synchronization risk and, as a consequence, delay us-

age of arbitrage opportunities. Rational arbitrageurs are supposed to know that the market will eventually collapse. They know that the bubble will burst as soon as a sufficient number of (rational) traders will sell out. However, the dispersion of rational arbitrageurs opinions on market timing and the consequent uncertainty on the synchronization of their sell-off are delaying this collapse, allowing the bubble to grow. In this framework, bubbles persist in the short and intermediate term because short sellers face synchronization risk, that is, uncertainty regarding the timing of the correction. As a result, arbitrageurs who conclude that other arbitrageurs are yet unlikely to trade against the bubble find it optimal to ride the still-growing bubble for a while. Instead of Abreu and Brunnermeier (2003)'s lack of synchronization from heterogeneous beliefs on the start of the bubble, Lin and Sornette (2011) emphasize that the lack of synchronization is more likely to result from the heterogeneous beliefs concerning the end of the bubble. Indeed, many reports both in the academic and professional literature state that sophisticated participants like hedge-funds correctly diagnosed the presence of a bubble and actually “surf” the bubbles, attracted by the potential large gains. Many reported that the largest uncertainty was how long it would continue its course (Gurkaynak, 2008; Sullivan, 2009).

The behavior of investors who are driven by group psychology, so-called *interacting agents*, and the aggregate behavioral outcomes, have also been studied using frameworks suggested by Weidlich and Haag (1983), Blume (1993; 1995), Brock (1993), Durlauf (1997; 1999), Kirman (1993), Brock and Durlauf (2000), Aoki and Yoshikawa (2007), Chiarella, Dieci and He (2009) and Hommes and Wagener (2009). Phan et al. (2004) summarize the formalism starting with different implementation of the agents' decision processes whose aggregation is inspired from statistical mechanics to account for social influence in individual decisions. Lux and Marchesi (1999), Brock and Hommes (1999), Kaizoji (2000, 2010), and Kirman and Teyssiere (2002) have developed related models in which agents' successful forecasts reinforce the forecasts. Such models have been found to generate swings in opinions, regime changes and long memory. An essential feature of these models is that agents are wrong for a fraction of the time but, whenever they are in the majority, they are essentially right by a kind of self-fulfilling prophecy. Thus, they are not systematically irrational (Kirman, 1997). Sornette and Zhou (2006) showed how irrational Bayesian learning added to the Ising model framework reproduces the stylized facts of financial markets. Harras and Sornette (2011) showed how over-learning from lucky runs of random news in the presence of social imitation may lead to endogenous bubbles and crashes.

Here, we develop a model of the pricing mechanism and resulting dynam-

ics of two co-existing classes of assets, a risky asset representing for instance the Internet sector during the dotcom bubble and a risk-free asset, in the presence of two types of investors having different opinions concerning the risky asset (Harrison and Kreps, 1978; Scheinkman and Xiong, 2003). The first type of traders is a group of rational investors who maximize their expected utility. The second type of traders is a group of “noise-traders” who trade only the risky asset by using heuristics such as past momentum and social imitation. The noise-traders do not consider the fundamentals, while the rational investor allocate their wealth based on their expectation of the future returns and risks of the risky asset.

Our model provides a natural explanation for the coexistence of intermediate horizon momentum and long horizon reversals in stock returns, that we can term the “bubble risk premium puzzle”. Our model also explains the intermittent and transient nature of bubbles, as well as their tendency to exhibit irregular cycles.

The main result of this paper is to demonstrate the existence of a bifurcation controlled by a key parameter ν , separating a stable regime from a bubble regime. In the stable regime, the rational investors are able to stabilize the price dynamics by their mean-reversal strategies based on fundamental analysis and utility maximization. In the bubble regime, the social imitation and momentum trading of the noise trader dominate and lead to intermittent bubbles.

The control parameter ν is found inversely proportional to the mean excess expected return of the risky asset anticipated by the rational traders. Thus, the larger the expectation of the growth of the risky asset, the more likely it is that the risky asset will be correctly priced. The control parameter ν is proportional to (i) the ratio of the population of noise to rational traders, (ii) the order sizes of noise traders, (iii) the rational expectation of the variance of the risky asset by the rational traders, and (iv) the risk aversion factor of the rational traders. Thus, the larger the fraction of noise traders and their market order, the larger is their impact and therefore the more likely it is for the risky asset to enter a bubble regime. Moreover, the rational agents have also an important role. If they anticipate a large variance of the returns of the risky stock and are risk adverse, then their demand will for the risky asset will be correspondingly low, allowing the noise traders to destabilize the price dynamics into the bubble regime.

The paper is organized as follows: the basic model is presented in Section 2 and 3, and analyzed theoretically in Section 4. Numerical simulations of the model are performed and the results are discussed in Section 5, together with a discussion and quantitative characterization of the price dynamics, its returns and momentum strategies during the dotcom bubble from 1998 to

2000. We conclude in Section 6.

2 Set-up of the model of an economy made of rational and of noise investors

We consider fixed numbers N_{rational} of rational investors and N_{noise} of noise investors who trade the same risky asset, represented here for simplicity by a single representative risky asset fund. The rational investors diversify between the risky asset and a risk-free asset on the basis of the maximization of their expected wealth utility, based on their expectation of the returns and variance of the risky asset over the next period. The noise traders use technical and social indicators, such as price momentum and social imitation to allocate their wealth. A dynamically evolving fraction of them buy the risky asset while others stay out of the market.

In the next subsection 2.1, we solve the standard allocation problem for the rational investors that determines their demand for the risky asset. Then, in subsection 2.2, the general ingredients controlling the dynamics of the demand of noise traders are developed.

2.1 Allocation equation for the rational investors

The objective of the N_{rational} rational investors is assumed to be the maximization at each time t of the expected utility of their expected wealth W_{t+1} at the next period (Chiarella et al., 2009; Hommes and Wagener, 2009). To perform this optimization, they select at each time t a portfolio mix of the risky asset and of the risk-free asset that they hold over the period from t to $t + 1$. Such one-period ahead optimization strategy can be reconciled with underlying expected utility maximizing stories (see, for example, Chiarella et al., 2002).

We assume that rational investors' preferences are characterized by the constant-absolute risk aversion (CARA) utility with risk aversion factor γ . The rational investors are assumed to be identical, so that we can consider the behavior of one representative rational investor hereafter. If the next period wealth is anticipated by the rational traders to be conditionally normal, as we shall assume here, then the maximization problem solved by the rational investor is known to be equivalent to the mean-variance optimization model, which means that only the expected portfolio value and its variance impact his allocation.

We denote P_t the price of the risky asset and X_t the number of the risky asset that the representative rational investor holds at instant t . We also

assume that the risky asset pays a dividend d_t at each period t . Similarly, P_{ft} and X_{ft} correspond to the price and number of a risk-free asset held by the rational agent. Thus, at time t , the wealth of the rational investor is given by

$$W_t = P_t X_t + P_{ft} X_{ft} . \quad (1)$$

The optimization by a rational agent of the CARA utility with risk aversion factor γ is equivalent to his optimization of the standard Expected Utility function

$$\text{EU}(W_{t+1}) := \text{E}_t[W_{t+1}] - \frac{\gamma}{2} \text{V}_t[W_{t+1}] , \quad (2)$$

where $\text{E}_t[W_{t+1}]$ is the expected value of the random variable corresponding to the wealth W_{t+1} at time $t+1$, while $\text{V}_t[W_{t+1}]$ is its variance, under the condition that W_t is known. The problem is to maximize $\text{EU}(W_{t+1})$ (2) by choosing the appropriate allocations X_t and X_{ft} at the current time t . To formulate it, the relation between the current wealth W_t and the future one W_{t+1} is needed. The wealth increment is equal to

$$W_{t+1} - W_t = (P_{t+1} - P_t)X_t + (P_{ft+1} - P_{ft})X_{ft} + d_{t+1}X_t . \quad (3)$$

This expression takes into account that the wealth at time $t+1$ is determined by the allocation choice at time t and the new values of the risky and the risk-free asset at time $t+1$, which include the payment of the dividend ($W_{t+1} = P_{t+1}X_t + P_{ft+1}X_{ft} + d_{t+1}X_t$).

Using the notations

$$x_t = \frac{P_t X_t}{W_t} , \quad R_{t+1} = \frac{P_{t+1}}{P_t} - 1 , \quad R_f = \frac{P_{ft+1}}{P_{ft}} - 1 , \quad (4)$$

we can rewrite (3) as

$$W_{t+1} = W_t + \left[R_f + x_t \left(R_{t+1} - R_f + \frac{d_{t+1}}{P_t} \right) \right] W_t . \quad (5)$$

It is easy to show that the conditional expectation and the variance of W_{t+1} given by (5) are equal to

$$\begin{aligned} \text{E}_t[W_{t+1}] &= W_t + R_f W_t + x_t \left(\text{E}_t[R_{t+1}] - R_f + \frac{\text{E}_t[d_{t+1}]}{P_t} \right) W_t , \\ \text{V}_t[W_{t+1}] &= W_t^2 x_t^2 \left(\sigma^2 + \frac{\text{Var}[d_{t+1}]}{P_t^2} \right) , \quad \sigma^2 = \text{Var}[R_{t+1}] , \end{aligned} \quad (6)$$

$\text{E}_t[R_{t+1}]$ (respectively σ^2) is the expected return (respectively variance) of the risky asset that is anticipated by the rational investors. $\text{E}_t[d_{t+1}]$ and

$\text{E}[d_{t+1}]$ and $\text{Var}[d_{t+1}]$ are respectively the mean and variance of the dividend paid at the next period, which are anticipated by the rational investors. These four quantities are taken as exogenous to the price dynamics developed below, because they reflect the information coming from the fundamental analysis performed by the rational investors. We thus obtain the following expression for the expected utility function

$$\text{EU}(W_{t+1}) = W_t + R_f W_t + x_t \left(\text{E}[R_{t+1}] - R_f + \frac{\text{E}_t[d_{t+1}]}{P_t} \right) W_t - \frac{\gamma}{2} W_t^2 x_t^2 \left(\sigma^2 + \frac{\text{Var}[d_{t+1}]}{P_t^2} \right). \quad (7)$$

The first-order condition applied to $\text{EU}(W_{t+1})$ with respect to the relative weight $x_t = \frac{P_t X_t}{W_t}$ of the risky asset in the portfolio leads to

$$P_t X_t = \frac{1}{\gamma \left(\sigma^2 + \frac{\text{Var}[d_{t+1}]}{P_t^2} \right)} \left(\text{E}_t[R_{t+1}] - R_f + \frac{\text{E}_t[d_{t+1}]}{P_t} \right). \quad (8)$$

Accordingly, the excess demand of the risky asset from $t-1$ to t of a rational investor, defined by

$$\Delta D_t^{\text{rational}} \equiv P_t X_t - P_{t-1} X_{t-1} = P_t X_t - \frac{P_t}{P_{t-1}} P_{t-1} X_{t-1}, \quad (9)$$

can be expressed as

$$\Delta D_t^{\text{rational}} = \frac{1}{\gamma \tilde{\sigma}^2} \left(\text{E}_t[R_{t+1}] - R_f + \frac{\text{E}_t[d_{t+1}]}{P_t} \right) - \frac{1}{\gamma \tilde{\sigma}^2} \frac{P_t}{P_{t-1}} \left(\text{E}_{t-1}[R_t] - R_f + \frac{\text{E}_t[d_t]}{P_{t-1}} \right), \quad (10)$$

where we define

$$\tilde{\sigma}^2 := \sigma^2 + \frac{\text{Var}[d_{t+1}]}{P_t^2}. \quad (11)$$

In the sequel, we assume that $\tilde{\sigma}^2$ is constant, independent of the price P_t . This corresponds to an expectation of the variance of dividends by the rational investors that tracks the (square of the) price. Alternatively, if $P_t \gg \sqrt{\frac{\text{Var}[d_{t+1}]}{\sigma^2}}$, $\tilde{\sigma}^2 \simeq \sigma^2$ and $\tilde{\sigma}^2$ is again approximately constant, as long as the rational investors form a non-varying expectation of the volatility of future prices of the risk asset.

In expression (10), $\text{E}_t[R_{t+1}] - R_f + \frac{\text{E}_t[d_{t+1}]}{P_t}$ represents the total excess expected rate of return of the risky asset from time t to $t+1$ above the risk-free rate. It is essential to remember that this term is not necessarily fixed but may vary with time t to reflect the continuous reassessment of the fundamental value of the risky asset by the rational agent. We thus define

$$\text{E}_t[R_{t+1}] - R_f + \frac{\text{E}_t[d_{t+1}]}{P_t} := \bar{A} \cdot (1 + \alpha_t), \quad (12)$$

where \bar{A} is the mean total expected return over the risk-free rate of the risky asset and α_t is a random variable with zero mean. In our numerical simulations below, we will consider the following specification for α_t ,

$$1 + \alpha_t = e^{\beta(z_t - \beta/2)}, \quad (13)$$

where $\{z_t\}$ is an i.i.d. sequence of Gaussian variables $\mathcal{N}(0, 1)$, and β is the parameter of the theory that quantifies the level of uncertainty in the formation of the rational agent anticipation. By construction from expression (13), $1 + \alpha_t$ is log-normally distributed with mean equal to 1, i.e., the mean of α is zero.

Using $P_t X_t = \frac{\bar{A}(1+\alpha_t)}{\gamma\tilde{\sigma}^2}$, which is derived from (8) and (12), the excess demand of the rational investors takes the form

$$\Delta D_t^{\text{rational}} = \frac{\bar{A}}{\gamma\tilde{\sigma}^2} \left(1 + \alpha_t - \frac{P_t}{P_{t-1}}(1 + \alpha_{t-1}) \right). \quad (14)$$

We will assume that $\bar{A} > 0$, so that the risky asset is desirable. This leads to a natural mean-reversing excess demand. Indeed, for instance if $P_t > P_{t-1}$, then the excess demand of the rational investors is on average negative and, symmetrically, if $P_t < P_{t-1}$, ΔD_t is on average positive. The rational investors thus tend to exert a stabilizing force on price by their demand that follows a negative feedback mechanism. On average, the excess demand of the rational investors

$$\mathbb{E}_t[\Delta D_t^{\text{rational}}] = -\frac{\bar{A}}{\gamma\tilde{\sigma}^2} \frac{P_t - P_{t-1}}{P_{t-1}} \quad (15)$$

is all the stronger, the larger is the average Sharpe ratio $\bar{A}/\tilde{\sigma}$ of the stock market performance of the risky asset, the smaller is the risk aversion coefficient γ , and the larger is the scaled realized return $(1/\tilde{\sigma})(P_t - P_{t-1})/P_{t-1}$ (realized return divided by the anticipated effective standard deviation of the risky asset defined by expressions (11) together with (6)). Expression (15) is equivalent to the standard result for the optimal asset demand of rational agents, when expressed in number of shares (see e.g. Chiarella, Dieci and He, 2009; Hommes and Wagener, 2009).

2.2 Excess demand of the noise traders

2.2.1 General framework

We assume that the noise traders are characterized by polarized decisions (in or out of the risky asset), that they tend to herd and that they

are trend-followers. A large body of literature indeed documents a lack-of-diversification puzzle as well as over-reactions. There is strong evidence for imitation and herding, even among sophisticated mutual fund managers. And technical analysis and chart trading is ubiquitous.

We account for the observations of lack-of-diversification by assuming that a noise trader is fully invested either in the risky asset or in the risk-free asset. In contrast with the rational agents, our noise traders have different opinions, which fluctuate stochastically according to laws given below. The number of noise investors invested in the risky asset (respectively invested in the risk-free asset) is $N_+(t)$ (respectively $N_-(t)$), and we have

$$N_+(t) + N_-(t) \equiv N_{\text{noise}} . \quad (16)$$

The excess demand of the noise traders over the time interval $(t-1, t)$ is equal to

$$\Delta D_t^{\text{noise}} = J \cdot (N_+(t) - N_+(t-1)) , \quad (17)$$

where J is the wealth allocated to financial investments per noise trader, assumed to be the same for all of them. In our simplified framework, we assume in addition that J is constant. In reality, it is a dynamical variable that also varies from noise trader to noise trader. The dynamics of J should be endogenously determined by the dynamics of the investment performance of noise traders. One can expect in general that the noise traders' wealths will become wildly distributed as a result of heterogenous investment decisions (Bouchaud and Mezard, 2000; Klass et al., 2007; Harras and Sornette, 2011). In turn, this will control a number of stylized facts of stock market returns, such as the fat tail distribution of return amplitudes. We leave this complication out of the present model formulation as we focus exclusively on the mechanisms for bubble formation.

Let us introduce the following variable

$$s_t = \frac{N_+(t) - N_-(t)}{N_{\text{noise}}} , \quad (18)$$

which can be interpreted as the bullish ($s_t > 0$) versus bearish ($s_t < 0$) stance of the noise traders with respect to the risky asset. We will refer to s_t as the bullish/bearish noise trader unbalance. With this definition (18) and (16), we have

$$N_+(t) = \frac{N_{\text{noise}}}{2}(1 + s_t), \quad N_-(t) = \frac{N_{\text{noise}}}{2}(1 - s_t) . \quad (19)$$

Therefore

$$N_+(t) - N_+(t-1) = \frac{N_{\text{noise}}}{2}(s_t - s_{t-1}) := \frac{N_{\text{noise}}}{2}\Delta s_t . \quad (20)$$

Expression (17) becomes

$$\Delta D_t^{\text{noise}} = \frac{1}{2} N_{\text{noise}} J \Delta s_t . \quad (21)$$

2.2.2 Master equation for the bullish/bearish noise trader unbalance Δs_t

Let us now specify the dynamics of Δs_t . We assume that, at each time step, each noise trader may change her mind and either sell her risky portfolio if she was previously invested or buy the risky portfolio if she had only the risk-free asset. Again, we assume an all-or-nothing strategy for each noise trader at each time step. Let $p_+(t-1)$ be the probability that any of the $N_+(t-1)$ noise traders who is currently (at time $t-1$) fully invested in the Internet portfolio decides to sell to remove her exposure during the time interval $(t-1, t)$. Analogously, let $p_-(t-1)$ be the probability that any of the $N_-(t-1)$ traders who are currently (at time $t-1$) out of the risky market decides to buy it. For a noise trader k who owns the risky asset, her specific decision is represented by the random variable $\zeta_k(p_+)$, which takes the value 1 (sell) with probability p_+ and the value 0 (keep the position) with probability $1 - p_+$. Similarly, for a noise trader j who does not own the risky asset, her specific decision is represented by the random variable $\xi_j(p_-)$, which takes the value 1 (buy) with probability p_- and the value 0 (remain invested in the risk-free asset) with probability $1 - p_-$. For given p_+ and p_- , the variables $\{\xi_j(p_-)\}$ and $\{\zeta_k(p_-)\}$ are i.i.d (identically independently distributed).

Aggregating these decisions over all noise traders yields the following master equation for the bullish/bearish noise trader unbalance Δs_t :

$$\Delta s_t = \frac{2}{N_{\text{noise}}} \left[\sum_{j=1}^{N_-(t-1)} \xi_j(p_-(t-1)) - \sum_{k=1}^{N_+(t-1)} \zeta_k(p_+(t-1)) \right] . \quad (22)$$

Using the i.i.d. property of the $\{\xi_j(p)\}$ and $\{\zeta_k(p)\}$ variables allows us to obtain the following exact expressions for the mean and variance of Δs_t :

$$\mathbb{E}[\Delta s_t] = p_-(t-1) \cdot (1 - s_{t-1}) - p_+(t-1) \cdot (1 + s_{t-1}) , \quad (23)$$

$$\text{Var}[\Delta s_t] = \frac{2}{N_{\text{noise}}} [p_-(t-1)(1 - p_-(t-1))(1 - s_{t-1}) + p_+(t-1)(1 - p_+(t-1))(1 + s_{t-1})] . \quad (24)$$

2.2.3 Influence of herding and momentum on the behavior of noise traders

As can be seen from (21) together with (22) and (23), the probabilities p_{\pm} embody completely the behavior of the noise traders. We assume that p_{\pm} at time $t - 1$ are both function of s_{t-1} (social imitation effect) defined by (18) and of a measure of price momentum H_t given by

$$H_t = \theta H_{t-1} + (1 - \theta) \left(\frac{P_t}{P_{t-1}} - 1 \right) , \quad (25)$$

which is nothing but the expression for an exponential moving averaging of the history of past returns. As usual, the parameter $0 \leq \theta < 1$ controls the length of the memory of past returns, the closer to 1, the longer the memory.

Considering that the probabilities p_{\pm} are functions of s_{t-1} and H_{t-1} ,

$$p_{\pm}(t - 1) = p_{\pm}(s_{t-1}, H_{t-1}) , \quad (26)$$

means that the noise traders make their decisions to buy or sell the risky Internet stock based on (i) the majority view held by their group and (ii) the recent capital gains that the Internet stock has provided over a time frame $\sim 1/(1 - \theta)$. We assume the noise traders buy and sell symmetrically with no bias: a strong herding in favor of the Internet stock or a strong positive momentum has the same relative effect on the drive to buy (or to sell) than a strong negative sentiment or strong negative momentum on the push to sell (or to buy). This is expressed by the following symmetry relation

$$p_{-}(s, H) = p_{+}(-s, -H) . \quad (27)$$

The simplest functions satisfying (27) are the linear expressions

$$p_{-}(s, H) = \frac{1}{2} [p + \kappa \cdot (s + H)] , \quad p_{+}(s, H) = \frac{1}{2} [p - \kappa \cdot (s + H)] , \quad (28)$$

that define two parameters p and κ . The positive parameter p controls the average holding time of the positions in the absence of any other influence. In other words, a position will last typically $\sim 2/p$ time steps in the absence of social imitation and momentum influence. The parameter κ quantifies the strength of social interactions and of momentum trading. For instance, for $\kappa > 0$, if there is already a majority of agents holding the Internet stock and/or if its price has been increasing recently, then the probability for noise traders holding the risk-free asset to shift to the risky Internet stock is increased and the probability for the noise traders who are already invested to sell their Internet stock is decreased. The reverse holds for $\kappa < 0$, which

describes “contrarian” traders. In the sequel, we will only consider the case $\kappa > 0$, which describes imitative and trend-following agents. Generalizations to allow for additional heterogeneous beliefs, involving mixtures as well as adaptive imitative and contrarian agents, is left for other communications. In this spirit, let us mention that Corcos et al. (2002) have introduced a simple model of imitative agents who turn contrarian when the proportion of herding agents is too large, which generates chaotic price dynamics.

Putting expressions (28) in (23) yields

$$\mathbb{E}[\Delta s_t] = -(p - \kappa)s_{t-1} + \kappa H_{t-1} , \quad (29)$$

$$\begin{aligned} \text{Var}[\Delta s_t] = \frac{2}{N_{\text{noise}}} & \left[p_-(t-1) \left(1 - \frac{p_-(t-1)}{2} \right) \right. \\ & \left. - \kappa \left(1 - \frac{p_-(t-1)}{2} \right) s(t-1)[s(t-1) + H(t-1)] + \frac{\kappa^2}{2} [s(t-1) + H(t-1)]^2 \right] . \end{aligned} \quad (30)$$

3 Dynamical market equations

3.1 Market clearing condition and price dynamics with noise traders as the only counterparts of the rational investors

The equation for the Internet stock price dynamics is obtained from the condition that the total excess demand summed over the rational and the noise traders vanishes:

$$N_{\text{rational}} \Delta D_t^{\text{rational}} + \Delta D_t^{\text{noise}} = 0 . \quad (31)$$

In other words, the net buy orders of noise traders are satisfied by the net sell orders of the rational traders, and vice-versa. Substituting in (31) expression (14) for the excess demands $\Delta D_t^{\text{rational}}$ of the representative rational investor and equation (21) for the excess demand $\Delta D_t^{\text{noise}}$ of the noise traders, we obtain

$$N_{\text{rational}} \frac{\bar{A}}{\gamma \tilde{\sigma}^2} \left(1 + \alpha_t - \frac{P_t}{P_{t-1}} (1 + \alpha_{t-1}) \right) + \frac{J}{2} N_{\text{noise}} \Delta s_t = 0 , \quad (32)$$

which yields the price equation

$$P_t = P_{t-1} \cdot \frac{1 + \alpha_t + \nu \Delta s_t}{1 + \alpha_{t-1}} . \quad (33)$$

Expression (33) shows that the price dynamics depends on two ingredients: (i) the varying components $\{\alpha_t\}$ of the rational investors' expectations of the Internet stock excess return and (ii) the time increments of the bullish/bearish noise trader unbalance $\{s_t\}$. The impact of $\{s_t\}$ is controlled by the value of the control parameter

$$\nu = \tilde{\sigma}^2 \frac{J\gamma}{2\bar{A}} \frac{N_{\text{noise}}}{N_{\text{rational}}} . \quad (34)$$

This parameter ν plays a crucial role in determining the price dynamics. All the interesting rich structures that we describe below result from ν taking non-zero values. In the opposite case where $\nu = 0$, which occurs in the absence of noise traders (or when they allocate zero wealth $J = 0$ to investing in the risky asset, which amounts to the same thing) or for vanishing risk aversion factor $\gamma = 0$ or zero expected volatility $\tilde{\sigma}^2 = 0$, the price dynamics corresponds to random fluctuations around a fundamental fixed price: $P_t = (1 + \alpha_t)P_0$. The fluctuation term α_t reflects the varying stochastic expectations of the rational traders. Note that this price dynamics obtained for $\nu = 0$ is fundamentally different from a geometric Brownian motion but is more like its increments.

The control parameter ν measures the relative influence of noise traders (the JN_{noise} term) with respect to the rational investors (the $2\bar{A}N_{\text{rational}}/\gamma\tilde{\sigma}^2$ term). The larger the number N_{noise} of noise traders and the larger their market order size J , the larger ν is. The influence of the noise traders is balanced by the rational traders, whose impact (making ν smaller) is all the larger, the larger their number N_{rational} , the larger their expectation \bar{A} of the mean excess return of the risky asset, and the smaller is their anticipation of the risks (variance $\tilde{\sigma}^2$) of the risky asset and their risk aversion γ .

3.2 Market clearing condition and price dynamics

However, there is a problem with condition (31) that leads to the price dynamics (33). Indeed, in absence of noise traders, $\nu = 0$, and as noted above, the solution of (33) is simply $P_t = (1 + \alpha_t)P_0$, assuming for simplicity that $\alpha_0 = 0$. This expression is nothing but the condition for $\Delta D_t^{\text{rational}}$ given by expression (14) to vanish. But this condition is quite artificial since it means that the price adjusts itself *without trading* (since there are no counterparty to the homogenous rational traders) in order to reflect their changing anticipation of the expected return of the risky asset. In absence of noise traders, there cannot be any trade and no price dynamics. This is nothing but a version of the no-trade theorem (Milgrom and Stokey, 1982) stating that no

profitable trade exists in markets in states of efficient equilibrium in the absence of noise traders or other non-rational interferences with prices and if information is common knowledge. A possible resolution of this paradox is to accept the fact that, in absence of noise traders, if the expectations formed by rational traders are constant ($\alpha_t = 0$ for all times), then the price is constant, which is consistent with the absence of trades. The problem remains however in the presence of time-varying rational traders' expectations.

The dynamics (33) of the price is an extension to this artificial condition that $\Delta D_t^{\text{rational}} = 0$ can be used to set the price dynamics in absence of noise traders. One could argue that noise traders are always present and the above no-trade problem is irrelevant. Actually, the resolution of the no-trade problem depends on the strategies used by the noise traders.

With the specification developed in subsection (2.2), leading in particular to the result (29) and (30) for the mean and variance of Δs_t , we have verified that the price always decays to zero asymptotically. This phenomenon, together with the no-trade phenomenon in the absence of noise traders, comes from the fact that our economy is closed and stationary (fixed number of traders and of the number of shares). A simple fix is to generalize the supply-demand equilibrium condition (31) into

$$N_{\text{rational}} \Delta D_t^{\text{rational}} + \Delta D_t^{\text{noise}} + \text{Supply}_t = 0 , \quad (35)$$

where Supply_t embodies the growth of the risky asset base representing our model economy, which could result for instance from the development of new products providing new market opportunities. For simplicity, we capture the growth of the economy by this external supply/demand contribution. This additional term can also be interpreted as the presence of a market maker (Chiarella, Dieci and He, 2009; Hommes and Wagener, 2009). For simplicity, we fix Supply_t to a constant value Supply .

Following step by step the derivation of the previous subsection, we find that the price dynamics (33) is changed into

$$P_t = P_{t-1} \cdot \frac{1 + \alpha_t + \delta + \nu \Delta s_t}{1 + \alpha_{t-1}} , \quad (36)$$

where

$$\delta = \frac{\gamma \tilde{\sigma}^2}{A} \cdot \text{Supply} . \quad (37)$$

As noted above, in absence of noise traders, the price is either constant for constant rational investors' expectations ($\alpha_t = 0$ for all times) or fluctuating like a random noise around a fundamental value. The price develops its usual geometric Brownian motion structure in the presence of noise traders,

as shown from the multiplicative structure of expression (36). In the limit where $\alpha_t = 0$ for all times, the price is determined by the product of terms such as $1 + \delta + \nu \Delta s_t$. In the limit where the decisions of noise traders are random (random Δs_t), expression (36) describe a pure geometric Brownian motion. It is the momentum-dependence of the strategies adopted by and the imitation between noise traders that provide a richer price dynamics, and in particular the emergence of bubbles for sufficiently large values of ν as we show below.

3.3 Complete set of dynamical equations

Let us put all ingredients of our model together to state concisely all the equations controlling the price dynamics coupled with the opinion forming process of the noise traders.

First, the price dynamics (33) must be combined with the equation for Δs_t . Using relations (19) and (22), the dynamics of Δs_t is given by the stochastic equation

$$\Delta s_t = \Delta s_t(s_{t-1}, H_{t-1}, p, \kappa), \quad (38)$$

where

$$\Delta s_t(s_{t-1}, H_{t-1}, p, \kappa) = \frac{2}{N_{\text{noise}}} \left[\sum_{k=1}^{\frac{N_{\text{noise}}}{2}(1-s_{t-1})} \xi_k(p_-(t-1)) - \sum_{k=1}^{\frac{N_{\text{noise}}}{2}(1+s_t)} \zeta_k(p_+(t-1)) \right]. \quad (39)$$

Combining expression (25) with (33) and summarizing all the above, we have

$$\begin{cases} P_t = P_{t-1} \cdot \frac{1 + \alpha_t + \delta + \nu \Delta s_t}{1 + \alpha_{t-1}}, \\ H_t = \theta H_{t-1} + (1 - \theta) \left(\frac{1 + \alpha_t + \delta + \nu \Delta s_t}{1 + \alpha_{t-1}} - 1 \right), \\ s_t - s_{t-1} := \Delta s_t = \Delta s_t(s_{t-1}, H_{t-1}, p, \kappa), \end{cases} \quad (40)$$

with $E[\Delta s_t]$ and $\text{Var}[\Delta s_t]$ given by expressions (29) and (30).

The set of equations (40) with (29,30) completely specify the model and its dynamics. The first equation in the set (40) expresses that the price dynamics is a kind of modified geometric Brownian process with multiplicative factors $\frac{1 + \alpha_t + \delta + \nu \Delta s_t}{1 + \alpha_{t-1}}$ reflecting the interplay between momentum trading, social imitation and rational pricing. The second equation in the set (40) shows the dynamics of the price momentum, which is one of the variables

influencing the decision of noise traders. The third equation in the set (40) describes the evolution of the majority opinions of the noise traders, who exhibit an imitative behavior with average and variance of the fluctuating components given by (29) and (30) that result from the flipping probabilities (28).

We have the following flow of causal influences:

1. Variations of noise traders' opinions and the changing anticipation of the fundamental Internet stock value by rational investors determine the next price change.
2. The sequence of past price changes determine the momentum H_t .
3. The noise traders' opinions evolve according to the momentum, while keeping a memory of their past values.

3.4 Control parameters and their time-scale dependence

The set of equations (40) depends on the following parameters:

1. β quantifies the level of uncertainty in the formation of the rational agents' anticipations (see expression (13)).
2. θ fixes the time scale over which noise traders estimate price momentum. By construction, $0 \leq \theta < 1$.
3. N_{Noise} is the number of noise traders that control the fluctuations of the majority opinion of noise traders.
4. p controls the average holding time of the positions of noise traders in the absence of any other influence.
5. κ quantifies the strength of social interactions and of momentum trading by noise traders. We will consider the regime $\kappa < p$.
6. δ ensures that the economy grows by external influx of resources, which reflects in stationary or growing stochastic prices.
7. ν is a combination of the 6 parameters $(\tilde{\sigma}^2, J, \gamma, \bar{A}, N_{\text{rational}}, N_{\text{Noise}})$, among which 5 of them $(\tilde{\sigma}^2, J, \gamma, \bar{A}, N_{\text{rational}})$ only appear in the theory via this reduced parameter ν (equation (34)). As we will see, all interesting structures that we describe below result from ν taking sufficiently large positive values.

In order to have an intuitive understanding of the role and size of these parameters, it is useful to discuss how they depend on the time scale over which traders reassess their positions. Until now, we have expressed the time t in units of a unit step 1, which could be taken for instance to be associated with the circadian rhythm, i.e., one day. But there is no fundamental reason for this choice and our theory has the same formulation under a change of the time step. Let us call τ the time interval between successive reassessments of the rational investors, with τ being measured in a calendar time scale, for instance, in seconds, hours or days.

First, the parameters N_{noise} and N_{rational} are a priori independent of τ , while they may be a function of time t . We neglect this dependence as we are interested in the dynamics over time scales of a few years that are characteristic of bubble regimes. The two parameters γ and J are also independent of τ .

In contrast, the parameters \bar{A} and $\tilde{\sigma}^2 = \sigma^2 + \frac{\text{Var}[d_{t+1}]}{P_t^2}$ are functions of τ , as both the average expected return and its variance depend on the time scale. As a result, the key parameter ν (34) of our theory is a priori in full generality a function of τ ($\nu = \nu(\tau)$). In some special cases, this dependence vanishes. This occurs for instance for the standard case where

$$\bar{A}(\tau) \sim \tau, \quad \tilde{\sigma}^2(\tau) \sim \tau, \quad (41)$$

for which the τ -dependence in expression (34) cancels out.

Similarly, when (41) holds, $\delta = \frac{\gamma \tilde{\sigma}^2}{\bar{A}}$. Supply has the dimension of Supply defined in expression (35), which is proportional to the time interval, hence $\delta \sim \tau$.

By construction, the parameter θ characterizing the memory of the price momentum influencing the decisions of noise traders depends on τ . This can be seen by replacing $t - 1$ by $t - \tau$ to make explicit the unit time scale in expression (25), giving

$$\frac{H_t - H_{t-\tau}}{\tau} = \frac{1 - \theta}{\tau} \left(\frac{P_t}{P_{t-\tau}} - 1 - H_{t-\tau} \right). \quad (42)$$

Requesting a bona-fide limit for small τ 's leads to

$$\frac{1 - \theta}{\tau} = \varrho = \text{const}, \quad (43)$$

where the time scale $\mathcal{T}_H := 1/\varrho$ is the true momentum memory. Thus, we have

$$1 - \theta = \varrho \cdot \tau, \quad \mathcal{T}_H := \frac{1}{\varrho} = \frac{\tau}{1 - \theta}. \quad (44)$$

4 Theoretical analysis and bifurcations

4.1 Reduction to deterministic equations

It is possible to get an analytical understanding of the solutions of the set of equations (40) if we reduce them into their deterministic components. The full set including their stochastic contributions will be studied below with the help of numerical simulations in the next section.

Taking $\alpha_t \equiv 0$ in (40) and replacing Δs_t (39) by its expectation $E[\Delta s_t]$ given by (29), we obtain

$$\begin{cases} P_t &= P_{t-1} \cdot (1 + \delta + \nu[(\kappa - p)s_{t-1} + \kappa H_{t-1}]), \\ H_t &= (1 - \theta)\delta + [\theta + (1 - \theta)\nu\kappa] H_{t-1} + (1 - \theta)\nu(\kappa - p)s_{t-1}, \\ s_t &= (1 + \kappa - p)s_{t-1} + \kappa H_{t-1}. \end{cases} \quad (45)$$

This system of three coupled deterministic equations has the following structure. The first equation has a bilinear structure and expresses the dependence of the price multiplicative process on the momentum H_t and on the average majority opinion s_t of the noise traders. The second and third equations constitute an autonomous system of two discrete time difference coupled linear equations expressed solely in terms of the variables H_t and s_t . This system has a unique fixed point

$$H^* = \delta, \quad s^* = \frac{\kappa\delta}{p - \kappa}. \quad (46)$$

Introducing the auxiliary variables

$$\tilde{s}_t := s_t - s^*, \quad \tilde{H}_t := H_t - H^*, \quad (47)$$

the second and third equations of the system (45) become

$$\begin{cases} \tilde{H}_t = [\theta + (1 - \theta)\nu\kappa]\tilde{H}_{t-1} + (1 - \theta)\nu(\kappa - 1)\tilde{s}_{t-1}, \\ \tilde{s}_t = (1 + \kappa - p)\tilde{s}_{t-1} + \kappa\tilde{H}_{t-1}. \end{cases} \quad (48)$$

4.2 Linear stability analysis

The stability of the fixed point (46) is determined by the eigenvalues λ defined by searching for solutions of the form

$$\tilde{H}_t = \lambda \cdot \tilde{H}_{t-1}, \quad \tilde{s}_t = \lambda \cdot \tilde{s}_{t-1}. \quad (49)$$

The corresponding characteristic equation for λ reads

$$(1 - \lambda + \kappa - p) [\theta - \lambda + (1 - \theta)\nu\kappa] - (1 - \theta)\nu(\kappa - p)\kappa = 0 , \quad (50)$$

whose solutions are

$$\lambda_{\pm}(\nu) = 1 + K\mu_{\pm} , \quad (51)$$

where

$$K = \sqrt{(1 - \theta)(p - \kappa)} , \quad (52)$$

and

$$\mu_{\pm}(\nu) = x \pm \sqrt{x^2 - 1} , \quad x = x(\nu) = \nu\kappa\rho - \rho - \frac{1}{4\rho} , \quad \rho = \frac{1}{2} \sqrt{\frac{1 - \theta}{p - \kappa}} . \quad (53)$$

We have assumed that $\kappa < p$, corresponding to the regime in which the strength of social interactions and of momentum trading by noise traders remains a perturbation to the evolution of their idiosyncratic opinions in the absence of any other influence.

The analysis of (51) with (53) shows the existence of several critical values for the key control parameter ν corresponding to the following conditions.

1. $x(\nu) < -1$: this corresponds to $\mu_{\pm}(x) < 0$ and thus $\lambda_{\pm}(\nu) < 1$, since $K := \sqrt{(1 - \theta)(p - \kappa)} < 1$.
2. $-1 \leq x(\nu) < 0$: $\lambda_{\pm}(\nu)$ acquires an imaginary part:

$$\lambda_{\pm}(\nu) = 1 + Kx(\nu) \pm iK\sqrt{1 - x^2} . \quad (54)$$

We have $0 < \text{Re}[\lambda_{\pm}(\nu)] < 1$, where $\text{Re}[z]$ denotes the real part of the complex number z .

3. $0 \leq x(\nu) < 1$: the expression for $\lambda_{\pm}(\nu)$ is again given by expression (54) but now $1 < \text{Re}[\lambda_{\pm}(\nu)]$,
4. $1 \leq x(\nu)$: $\lambda_{\pm}(\nu)$ is given by

$$\lambda_{\pm}(\nu) = 1 + K \left[x \pm \sqrt{x^2 - 1} \right] . \quad (55)$$

Both eigenvalues $\lambda_{\pm}(\nu)$ are larger than 1.

These four regimes are delineated by three critical values of ν respectively given by

$$\nu_{-1} = \frac{1}{\kappa\rho} \left(\rho + \frac{1}{4\rho} - 1 \right) \quad \leftrightarrow \quad x(\nu_{-1}) = -1 , \quad (56)$$

$$\nu_0 = \frac{1}{\kappa\rho} \left(\rho + \frac{1}{4\rho} \right) \quad \leftrightarrow \quad x(\nu_0) = 0 , \quad (57)$$

$$\nu_1 = \frac{1}{\kappa\rho} \left(\rho + \frac{1}{4\rho} + 1 \right) \quad \leftrightarrow \quad x(\nu_1) = 1 . \quad (58)$$

We thus have the following classification of stability regimes.

1. For $0 \leq \nu < \nu_{-1}$, $\lambda_{\pm}(\nu) < 1$, In this case, the fixed point (H^*, s^*) (46) is absolutely stable. Starting for arbitrary initial values of the momentum and of the average noise trader opinion, they will converge exponentially monotonously to the values (46), corresponding to a price growing exponentially with rate δ :

$$P_t = P_{t-1}(1 + \delta) \quad \Rightarrow \quad P_t \simeq P_0(1 + \delta)^t \sim P_0 e^{\delta t} , \quad (59)$$

Reintroducing the stochastic terms $\{\alpha_t\}$ and $\{\Delta s_t\}$ gives rise to a price dynamics not too far from a standard geometric Brownian motion with drift δ . For instance, in the case $\nu = 0$, $x(0) = -\rho - \frac{1}{4\rho} \leq -1$ giving $\lambda_+ = 1 - K/2\rho$ and $\lambda_- = 1 - 2\rho K$, which are both between 0 and 1.

2. For $\nu_{-1} < \nu < \nu_0$, the eigenvalues λ_{\pm} develop an imaginary part but keeps their real part $0 < \text{Re}[\lambda_{\pm}(\nu)]$ smaller than 1. The fixed point (H^*, s^*) (46) is still stable but perturbations return to it by following exponential damped oscillations.
3. For $\nu_0 < \nu < \nu_1$, the eigenvalues λ_{\pm} have again an imaginary part but their real part $\text{Re}[\lambda_{\pm}(\nu)]$ becomes larger than 1. The fixed point (H^*, s^*) (46) becomes unstable and perturbations grow with an exponential envelop with an oscillatory behavior. This regime corresponds to recurrent bubbles.
4. For $\nu_1 < \nu$, both eigenvalues λ_{\pm} are real and larger than 1. The fixed point (H^*, s^*) (46) is unstable and solutions grow exponentially until they saturate to their limiting values. This regime corresponds to a single explosive bubble.

This stability analysis suggests that significant deviations in the form of transient explosive price trajectories characterize what can be called “bubbles” that occur for values of the key parameter ν larger than ν_0 given by expression (57). For $\nu_{-1} < \nu < \nu_0$, we should expect significative deviations of the properties of the price dynamics from those of a pure geometric Brownian motion, in particular due to the interplay between the stochastic terms $\{\alpha_t\}$ and $\{\Delta s_t\}$ and the oscillatory eigenvalues. For $\nu > \nu_0$, we also expect

that a different behavior of the price dynamics depending on whether ν is smaller or larger than ν_1 . For $\nu_0 < \nu < \nu_1$, bubble regimes should occur with a quasi-periodic behavior, perturbed by the stochastic terms $\{\alpha_t\}$ and $\{\Delta s_t\}$. For $\nu > \nu_1$, isolated explosive bubbles should occur.

Expressions (56-58) show that ν_{-1}, ν_0 and ν_1 are decreasing functions of κ and ν_0 is also a decreasing function of $\rho := (1 - \theta)/\tau$. Thus, the larger is the strength κ of social interactions and of momentum trading by noise traders, the smaller are the bifurcation values ν_{-1}, ν_0 and ν_1 . Similarly, the shorter the time scale $\tau/(1 - \theta)$ over which the momentum is calculated, the smaller is the key bifurcation value ν_0 . Hence, the transition to a bubble regime is favored by noise traders who are more social, and who use more momentum trading with shorter time horizons.

4.3 Time scales of price and bubble dynamics

For the regimes where oscillations occur in the deterministic treatment, we can estimate the characteristic periods by returning to the time-scale analysis of subsection 3.4. In terms of the time interval τ between successive decisions of rational investors, for small τ 's, we have

$$p = \chi \cdot \tau, \quad \kappa = \varkappa \cdot \tau, \quad (60)$$

where χ and \varkappa are constant rates governed by the random Poisson-like process of noise trading. Since $1 - \theta \sim \tau$ according to (43), the parameter ρ defined in (53) is independent of τ but, by (56-58), the critical values ν_{-1}, ν_0 and ν_1 have all the dimension of $1/\kappa$, hence

$$\nu_{-1} \sim \tau^{-1}, \quad \nu_0 \sim \tau^{-1}, \quad \nu_1 \sim \tau^{-1}. \quad (61)$$

This dependence (61) implies that the bubble regime will be more probable at large time scales since the critical value ν_0 is a decreasing function of τ . With the above time-scaling relationships, one can even state that bubbles are unavoidable in our model at sufficiently long times as soon as ν is non vanishing, i.e., when there are some noise traders who enter the market. Indeed, for a fixed ν , there always exists a sufficiently large τ such that $\nu_0(\tau)$ becomes smaller than ν .

Expression (51), (54) and (55) show that the parameter K defined by (52) sets the typical time scale \mathcal{T}_0 of the dynamics, either pure exponential damping, oscillatory exponential damping, oscillatory exponential growth or pure exponential growth. It is given by the following expression

$$\mathcal{T}_0 \simeq \tau \frac{2\pi}{\sqrt{\varrho(\chi - \varkappa)}}. \quad (62)$$

5 Numerical simulations and comparison with the dotcom bubble

5.1 Estimation of parameter values

Let us take $\tau = 1$ day and assume a typical memory used by noise traders for the estimation of price momentum equal to about one month. This amounts approximately to 20 trading days, hence $\mathcal{T}_H \simeq \frac{\tau}{1-\theta} = 20$, leading to $\theta = 0.95$.

For the parameter p in expressions (28) equal to twice the probability that during a given day some noise trader will buy (or sell) the risky asset, we posit $p = 0.2$. In other words, the natural trading frequency of traders in absence of social influence is about one week. For the parameter κ in (28) describing the strength of social interactions and of momentum trading, we assume that it is close to the value of parameter p , specifically, $\kappa = 0.95p = 0.19$. According (62), this leads to a characteristic time scale $\mathcal{T}_0 \simeq 280$ days. With the above values, we have $\rho = 1.12$, leading to $\nu_{-1} \simeq 1.88$, $\nu_0 \simeq 6.58$ and $\nu_1 \simeq 11.3$.

Let us assume a typical value $\bar{A} \simeq 0.1\%$ for the average return per day of the Internet stock expected by the rational investors and let us assume that $J \cdot \gamma \simeq 1$. This last assumption amounts to set the scale of the price of the Internet stock to be of the order of unity in the absence of bubbles. Expression (34) leads to

$$\nu \simeq 10^3 \cdot \tilde{\sigma}^2 \cdot \frac{N_{\text{noise}}}{2N_{\text{rational}}}. \quad (63)$$

The expected variance of the Internet stock that is anticipated by the rational investors is typically of the order $\tilde{\sigma}^2 \simeq 0.001$, which leads to $\nu \simeq 0.25N_{\text{noise}}/N_{\text{rational}}$. For these parameters, we see that the bubble regime occur only when $N_{\text{noise}} \simeq 25N_{\text{rational}}$, i.e., noise traders constitute more than about 96% of all traders. This is not unrealistic as it is well documented that, at certain times, even diehard value investment houses turn to technical analysis in order to “surf” the bubble (Andreassen and Kraus, 1990; Case and Shiller, 1988; Frankel and Froot, 1988). Lastly, we consider an effective number of noise traders equal to $N_{\text{noise}} \simeq 10^3$. This relatively small number should not be misleading, as it is well-documented that prices are moved by a small number of investors (Coyne and Witter, 2002). And we assume that $\beta = 0.3$. Finally, we choose the parameter δ , for any given ν , such that the price does not grow on average over very long times, even if it is punctuated by significant bubbles at intermediate time scales.

Summarizing, the numerical simulations presented in the figures correspond to

$$\theta = 0.95, \quad p = 0.2, \quad \kappa = 0.19, \quad \beta = 0.3, \quad N_{\text{noise}} = 10^3 \quad (64)$$

for which the critical values of the key parameter ν are $\nu_{-1} \simeq 1.88$, $\nu_0 \simeq 6.58$ and $\nu_1 \simeq 11.3$.

5.2 Results and interpretation

Figures 1-4 show the time dependence of the variables s_t , H_t and P_t that are generated by numerical solutions of the set (40) of three equations for four different values of the control parameter ν , which has been found to determine the different price dynamic regimes according to the stability analysis presented in section 4.

Figure 1 corresponds to $\nu = 1 < \nu_{-1} \simeq 1.88$, for which the fixed point (H^*, s^*) (46) is linearly stable in the absence of stochastic terms. As expected, one can observe that the price fluctuates close to the fundamental value.

Figure 2 corresponds to $\nu = 5$ ($\nu_{-1} < \nu < \nu_0$), i.e., above the first bifurcation value $\nu_{-1} \simeq 1.88$ but slightly below the second bifurcation point $\nu_0 \simeq 6.58$. One can observe a much stronger persistence of s_t and H_t compared with figure 1. Correspondingly, the price P_t starts to exhibit significant deviations from its base level, in the form of intermittent bursts of relatively moderate amplitudes compared with the plots shown in figures 3 and 4.

Figure 3 corresponds to $\nu = 8$ ($\nu_0 < \nu < \nu_1$), i.e., above the bifurcation value $\nu_0 \simeq 6.58$ for which the fixed point (H^*, s^*) (46) becomes linearly unstable, but below the last bifurcation point $\nu_1 \simeq 11.3$. One can observe that the variable s_t , which captures the bullish ($s_t > 0$) versus bearish ($s_t < 0$) stance of the noise traders with respect to the risky asset, exhibits spells of stable values corresponding to strong bullish opinions. The breakdown of the symmetry $s_t \rightarrow -s_t$ is due to the parameter δ (37) that embodies the growth of the overall value of the risky asset as reflected for instance by an increasing number of shares.

Figure 4 corresponds to $\nu = 15$ ($\nu > \nu_1$), i.e., above the last bifurcation point $\nu_1 \simeq 11.3$, for which the dynamics is very unstable. Figure 4 illustrates that, due to the herding effect quantified by the variable s_t , there is significant correlation between normalized excess s_t and price P_t , giving rise to bubble-to-crash like behavior, as expected from the theoretical stability analysis of section 4. Note the extraordinary large amplitudes of the prices in this last “wild” regime.

5.3 Comparison with the dotcom bubble

This section compares the insights obtained from the above theoretical and numerical analyses to empirical evidence on momenta and reversals in the period when the dotcom bubble developed¹. We study the characteristics of the share prices of Internet-related companies over the period from January 1, 1998 to December 31, 2002, which covers the period of the development of the dotcom bubble and its collapse. We use the list of 400 companies belonging to the Internet-related sector that has been published by Morgan Stanley and has already been investigated by Ofek and Richardson (2003). The criteria for a company to be included in that list is that it must be considered a “pure” internet company, i.e., whose commercial goals are associated exclusively to the Internet. This implies that technology companies such as Cisco, Microsoft, and telecommunication firms, with extensive Internet-related businesses, are excluded.

Figure 5 graphs the index of an equally weighted portfolio of the Internet stocks over the sample period of January 1998 to December 2002. Figure 5 is strikingly similar to the dynamics generated by the theoretical model in the bubble regime shown in figure 3. The time evolution of the equally weighted portfolio of the Internet stocks is strikingly different from that shown in figure 6 for the index of an equally weighted portfolio of non-Internet stocks over this same period. The two indexes are scaled to be 100 on January 2, 1998. The two figures illustrate clearly the widely held view that a divergence developed over this period between the relative pricing of Internet stocks and the broad market as a whole. In the two year period from early 1998 through February 2000, the internet related sector earned over 1300 percent returns on its public equity while the price index of the non-internet sectors rose by only 40 percent. However, these astronomical returns of the Internet stocks had completely evaporated by March 2001.

Annual Returns for Internet and non-Internet stock indices

Year	1998	1999	2000	2001	2002
Internet stock index	116.8%	815.6%	−875.9%	−62%	−48.8%
(per month)	(9.7%)	(68%)	(−73%)	(−5.2%)	(−4.1%)
Non-internet stock index	6.5%	17%	−9%	3.6%	−9%
(per month)	(0.5%)	(1.4%)	(−0.8%)	(0.3%)	(−0.7%)

¹The dotcom bubble (followed by its subsequent crash) is widely believed to be a speculative bubble, as documented by Ofek and Richardson (2003), Brunnermeier and Nagel (2004), and Battalio and Schultz (2006).

We now focus our attention on the profitability of the momentum strategies studied by Jegadeesh and Titman (1993, 2001) and others. Table 1 provides some descriptive statistics about annual returns of the Internet-stock index versus of the non-Internet stock index from the beginning of 1998 to the end of 2002. In the 12 months of 1998, the annual cumulative return of the Internet stock index was 117 percent, while that of the non-Internet stock index was 6.5 percent. In the 12 months of 1999, the annual cumulative return of the Internet stock index surged to 816 percent, and that of the non-Internet stock index increased to 16.6 percent. The Internet stock index clearly outperformed the non-Internet stock index by 800 percent in 1999. This implies a strong profitability of momentum strategies applied to the Internet stocks over the period of the dotcom bubble. However, after its burst in March 2000, the return of the Internet stocks sharply declined, from 2000 to 2002. In the 12 months of 2000, the annual return of the internet-stock index fell to - 876 percent, followed by - 62 percent and - 49 percent in 2001 and in 2002, respectively. On the other hand, the annual returns of the non-Internet stock index in the period from 2000 to 2002 remain modest in amplitude at - 9 percent, 3.6 percent and - 9 percent, respectively. After the bust of the dotcom bubble, the Internet stocks continued to underperform the non-Internet stocks.

Table 2 shows the cumulative returns for the Internet stock index and for the non-Internet stock index in the five years from the beginning of 1998 to the end of 2002. The cumulative return of the Internet stock index in the first 24 months of the holding period is 932.5 percent, but the cumulative returns ends at the net loss of - 54.2 percent over the five year holding period. In contrast, the cumulative returns of the non-Internet stock index over the same five year holding period is 8.6 percent.

Cumulative Returns for Internet and non-Internet stock indices

Year	1998	1999	2000	2001	2002
Internet stock index	116.8%	932.5%	56.6%	-5.4%	-54.2%
(per month)	(9.7%)	(38.9%)	(1.6%)	(-0.1%)	(-0.9%)
Non-internet stock index	6.5%	23.1%	-14%	17.6%	8.6%
(per month)	(0.5%)	(1.0%)	(0.4%)	(0.4%)	(0.1%)

In summary, these empirical facts constitute strong evidence for the Internet stock for momentum profit at intermediate time scales of about two years and reversals at longer time scales of about 5 years. These empirical

facts confirm for this specific bubble and crash period general evidence documented by many researchers (e.g. Jegadeesh and Titman, 1993, 2001). They are consistent with the stylized facts described by the model that predict that the momentum profits will eventually reverse in cycle bubbles and crashes as illustrated above.

6 Conclusions

We have introduced a model of financial bubbles with two assets (risky and risk-less), in which rational investors and noise traders co-exist. Rational investors form continuously evolving expectations on the return and risk of a risky asset and maximize their expected utility with respect to their allocation on the risky asset versus the risk-free asset. Noise traders are subjected to social imitation and follow momentum trading. We have found the existence of a set of bifurcations controlled by the relative influence of noise traders with respect to rational investors that separate a normal regime of the price dynamics to a phase punctuated by recurrent exponentially explosive bubbles.

To the important question of whether and when rational investors are able to stabilize financial markets by arbitraging noise traders, we found that the answer is not clear-cut but rather reveal the existence of nonlinear changes of regimes controlled by a unique control parameter ν , which is a reduced combination of six characteristics of the rational and noise agents. For ν smaller than a critical bifurcation value, rational investors are able to stabilize the price dynamics by their mean-reversal strategies. Above the bifurcation point, the noise traders' social imitation and momentum trading dominate and give rise to different types of bubbles and crashes.

The control parameter ν is inversely proportional to the mean excess expected return of the risky asset anticipated by the rational traders and is proportional to (i) the ratio of the population of noise to rational traders, (ii) the order sizes of noise traders, (iii) the rational expectation of the variance of the risky asset by the rational traders, and (iv) the risk aversion factor of the rational traders.

The model has been found to account well for the behavior of traders and for the price dynamics that developed during the dotcom bubble in 1995-2000. Momentum strategies have been shown to be transiently profitable, supporting the hypothesis that these strategies enhance herding behavior.

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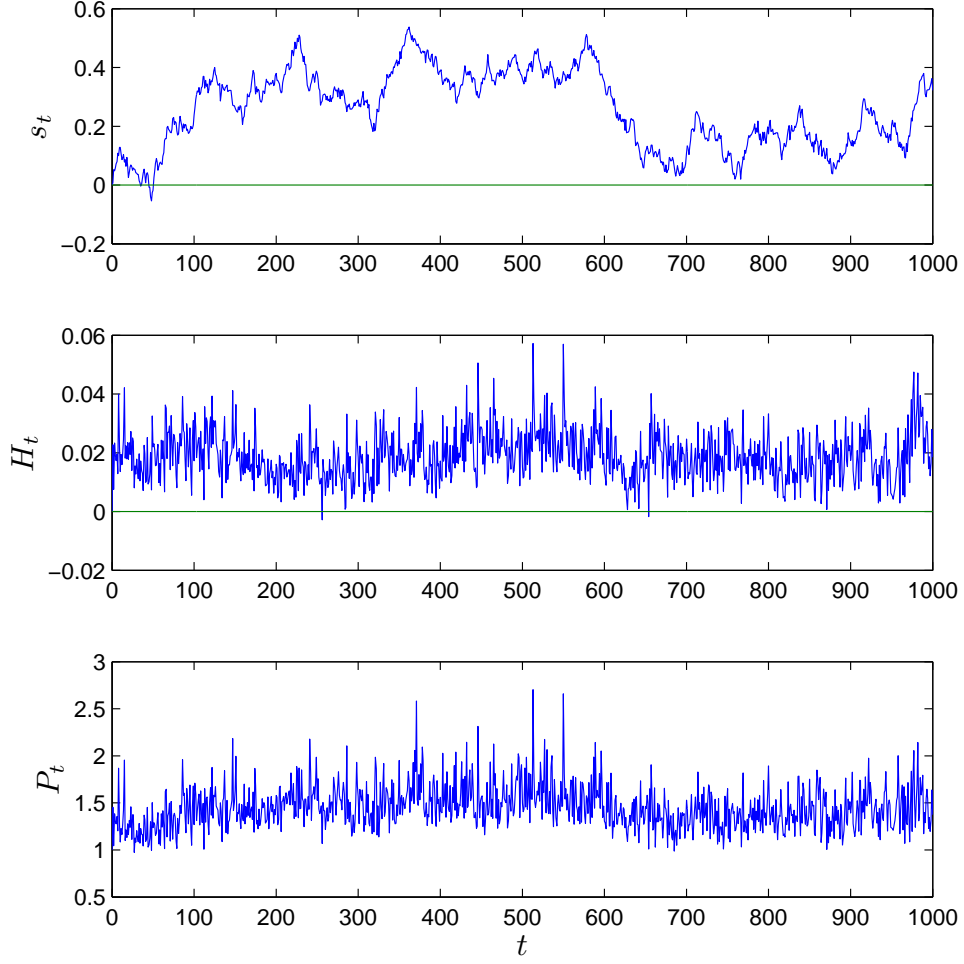


Fig. 1: Plot of s_t , H_t , and price P_t in the case of parameters (64) and for $\nu = 1$ and $\delta = 10^{-4}$. Stable price regime fluctuating around the fundamental price with no bubble.

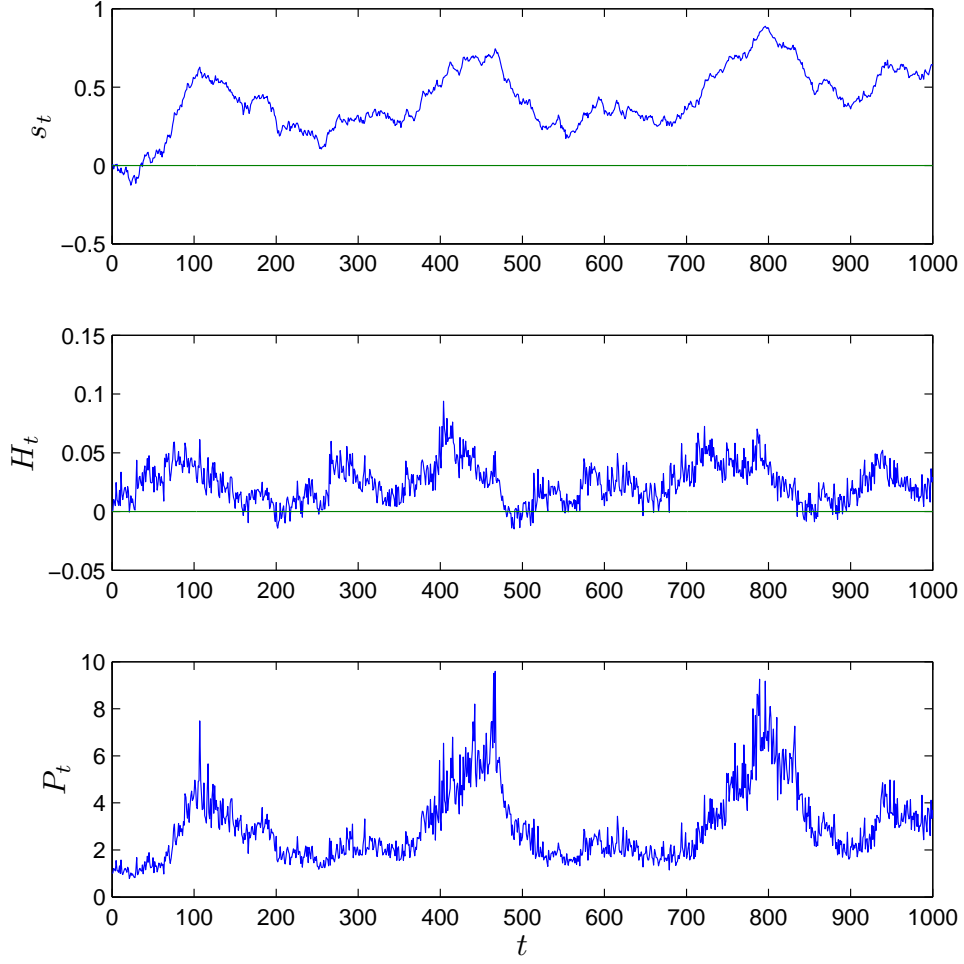


Fig. 2: Plot of s_t , H_t , and price P_t in the case of parameters (64) and for $\nu = 5$ and $\delta = 1.85 \cdot 10^{-3}$.

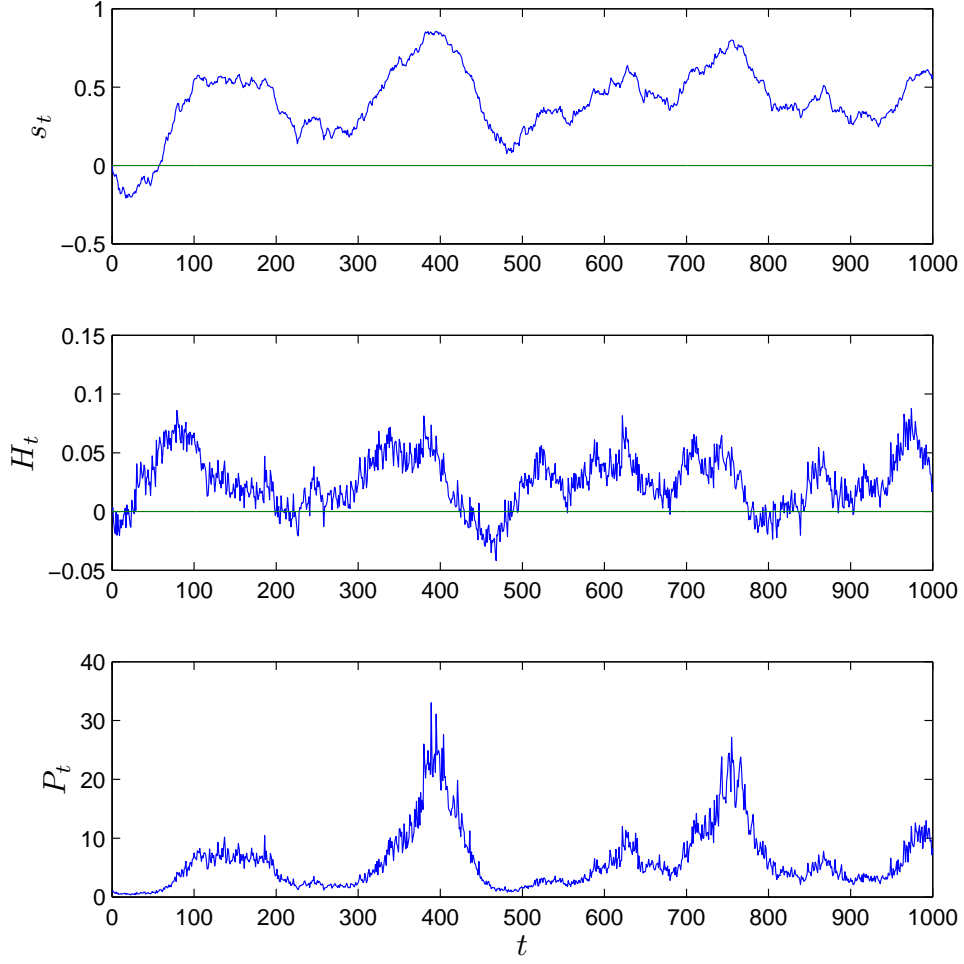


Fig. 3: Plot of s_t , H_t , and price P_t in the case of parameters (64) and for $\nu = 8$ and $\delta = 4.7 \cdot 10^{-3}$.

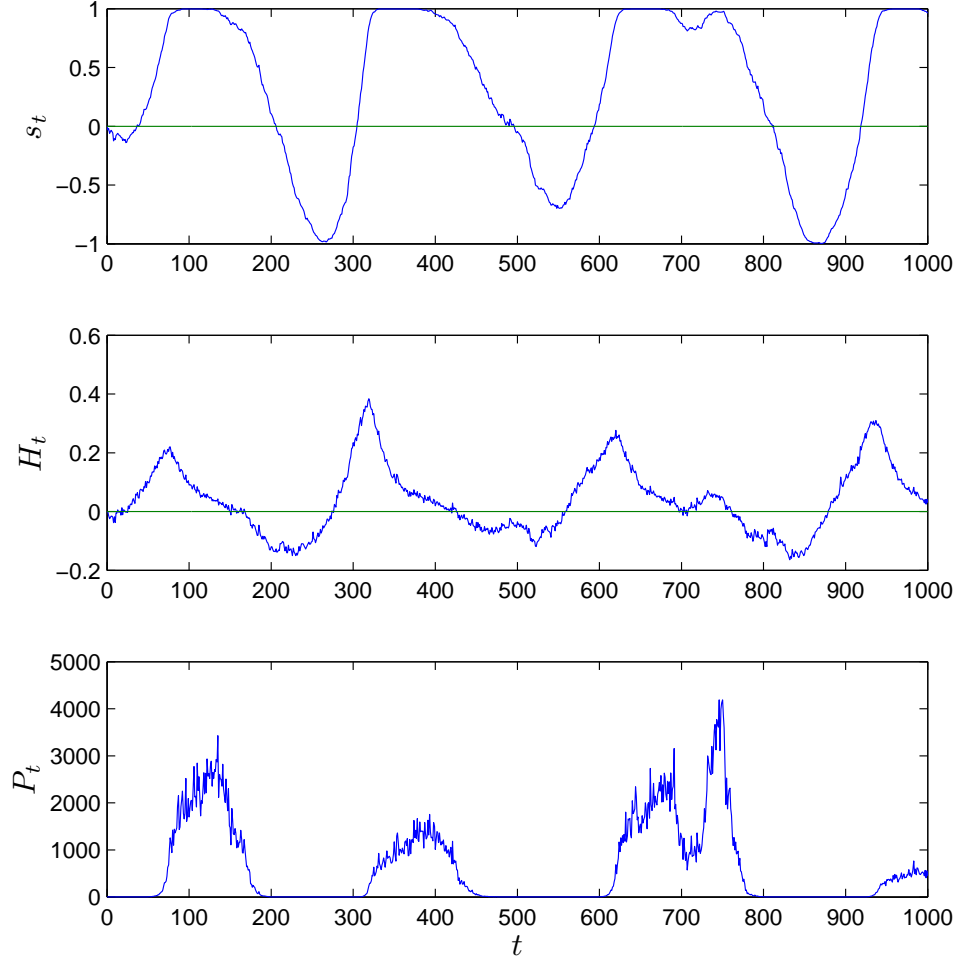


Fig. 4: Plot of s_t , H_t , and price P_t in the case of parameters (64) and for $\nu = 15$ and $\delta = 2.4 \cdot 10^{-2}$

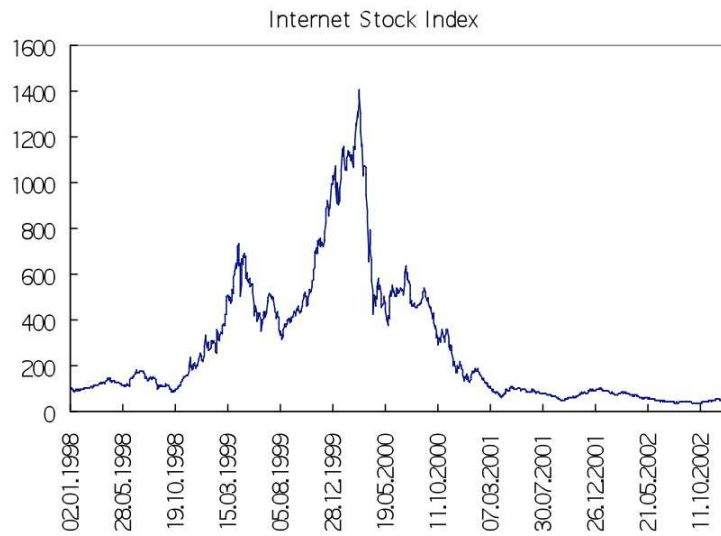


Fig. 5: The equally weighted Internet stock index for the period 1/2/1998-12/31/2002. The index is scaled to be 1 on 1/2/1998.

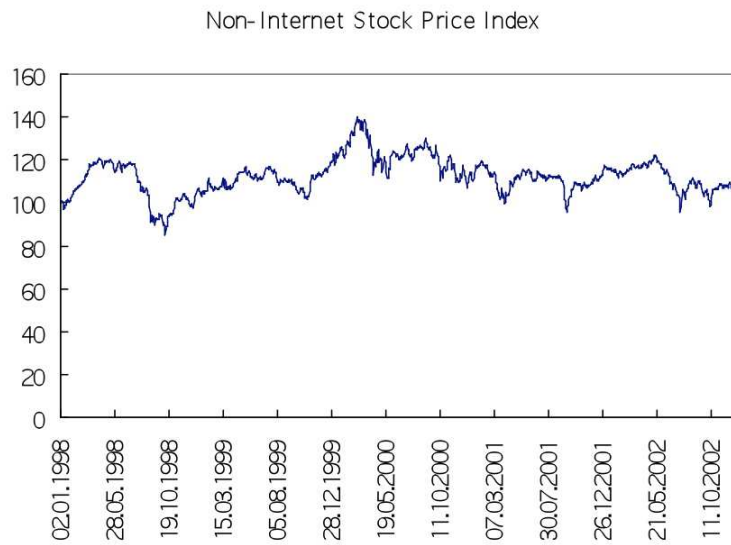


Fig. 6: The equally weighted non-Internet stock index for the period 1/2/1998-12/31/2002. The index is scaled to be 1 on 1/2/1998.